JOURNAL OF APPROXIMATION THEORY **62**, 350–357 (1990)

An Extension of the Fusion Lemma

GERALD SCHMIEDER*

Mathematisches Institut, Universität Würzburg, Am Hubland, D-8700 Würzburg, West Germany

Communicated by P. L. Butzer

Received April 18, 1988; revised October 12, 1988

1. INTRODUCTION

In the theory of approximation by rational functions in the complex plane $\mathbb{C}$, the following Fusion Lemma of A. Roth plays an important role (cf. [6; 2, p. 113 ff.]; for approximation on Riemann surfaces see [7, 8]):

For every pair of disjoint compact sets K_1, $K_2 \subset \mathbb{C}$ there is a constant $\alpha = \alpha(K_1, K_2)$ with the property: For arbitrary rational functions r_1, r_2 and any given compact set $k \subset \mathbb{C}$, there is some rational function r with

$$|r(z) - r_j(z)| \leqslant \alpha \cdot \sup_{w \in k} |r_1(w) - r_2(w)|$$

for all $z \in K_j \cup k$, $j = 1, 2$.

In [2, p. 116] the question has been posed to what extent the assumption $K_1 \cap K_2 = \varnothing$ in the Fusion Lemma can be replaced by a weaker condition.

It has been pointed out by P. M. Gauthier that the Fusion Lemma is not true for $K_1 \cap K_2 \neq \varnothing$ in general. Even for rectangles K_1, K_2 the Lemma becomes false if there is a common edge of K_1 and K_2, as D. Gaier [3] has shown. The statement remains true if $k := K_1 \cap K_2$ is a finite set and $\mathbb{C} \setminus (K_1 \cup K_2)$ has only finitely many components (see [3]).

In this paper we will prove a more general extension of the Fusion Lemma which guarantees the existence of the desired function r in several cases in which $K_1 \cap K_2$ is a continuum. Moreover the compact set k can be chosen arbitrarily, so that $k = K_1 \cap K_2$ is not required.

The result is a combination of the classical Fusion Lemma [6] and the Lemma of Nersesjan [5] (see Section 3). Actually between both there

* Present address: Fachbereich 6 Mathematik, Universität Oldenburg, Postfach 2503, D-2900 Oldenburg, West Germany.

0021-9045/90 $3.00

seems to be a surprising relation: where in this paper the Fusion Lemma is improved by using the Lemma of Nersesjan this extended Fusion Lemma again leads to an improved version of Nersesjan's Lemma which we will study in a forthcoming paper. On the other hand the "best-possible" (if true) version of Nersejan's Lemma (Section 3, Remark 6) would give the best-possible version of the Fusion Lemma.

2. The Main Result

For a set $M \subset \mathbb{C}$ and $\delta > 0$ let $M_\delta := \{z \in \mathbb{C} \mid \operatorname{dist}(z, M) \leqslant \delta\}$ where dist denotes the Euclidean distance.

DEFINITION. Let A, $B \subset \mathbb{C}$ be compact sets. We say that A is *extensible relative to B* iff there is some $\delta_0 > 0$ such that for all $\delta > 0$, $\delta < \delta_0$ there is a compact set C $(= C(\delta))$ with

(i) $(A \setminus B_{2\delta})_\delta \cup A \subset C \subset \mathbb{C} \setminus \mathring{B}$,

(ii) $\partial C \cup \partial B \subset \partial(C \cup B)$,

(iii) $\mathbb{C} \setminus (C \cup B)$ consists only of a finite number of components.

Remark. In Fig. 1 the set K_1 is extensible relative to K_2 but not in Fig. 2. In both examples K_2 is extensible relative to K_1.

Our main result is the following extension of the Fusion Lemma:

THEOREM. *Let K_1, $K_2 \subset \mathbb{C}$ be compact sets such that $\mathbb{C} \setminus (K_1 \cup K_2)$ has only finitely many components and assume that $\bar{\mathring{K}}_1$ is extensible relative to $\bar{\mathring{K}}_2$, or that $\bar{\mathring{K}}_2$ is extensible relative to $\bar{\mathring{K}}_1$. Then there exists some constant $\alpha = \alpha(K_1, K_2)$ such that for arbitrary rational functions r_1, r_2 and any*

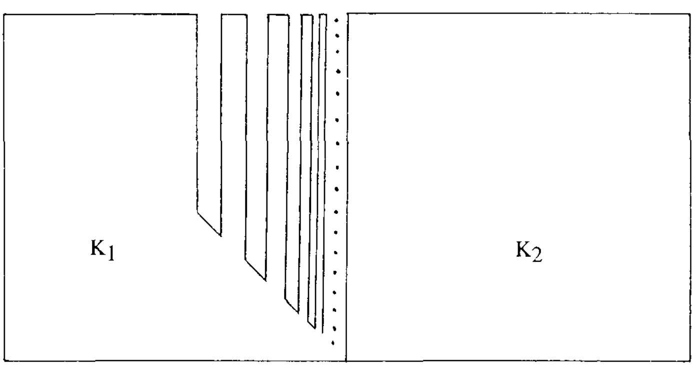

FIGURE 1

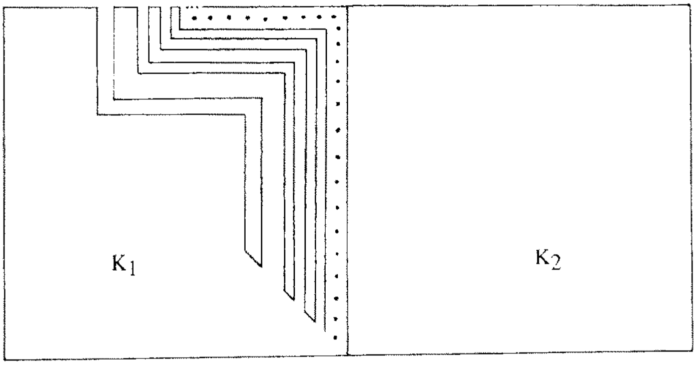

FIGURE 2

given compact set $k \subset \mathbb{C}$ *there is some rational function* r *with* $|r(z) - r_j(z)| \leqslant \alpha \cdot \sup\{|r_1(w) - r_2(w)| \mid w \in k \cup (K_1 \cap K_2)\}$ *for all* $z \in K_j \cup k$, $j = 1, 2$.

We shall give the proof in Section 4. Note that the assumption that $\mathbb{C} \setminus (K_1 \cup K_2)$ consists only of finitely many components could be made in the classical Fusion Lemma without any loss of generality by suitable enlargement of K_1 and K_2.

3. Remarks about the Lemma of Nersesjan

The following Lemma is due to Nersesjan ([5], cf. [2, p. 143]). The role it plays in tangential approximation is similar to that played by the Fusion Lemma in uniform approximation.

Lemma. *Let* $F \subset \mathbb{C}$ *be a compact set and suppose that* $\mathbb{C} \setminus F$ *is the union of only a finite number of components. Given an open set* $G \subset F$ *with* $\partial G \subset \partial F$ *(i.e., G is a union of components of* $\mathring{F}$*) and* $\varepsilon > 0$ *there is a rational function* R *with*

(a) $|R(z)| < \varepsilon$ *for* $z \in G \setminus (\partial G)_\varepsilon$

(b) $|R(z) - 1| < \varepsilon$ *for* $z \in F \setminus G_\varepsilon$

(c) $|R(z)| < c$ *for* $z \in F$, *where* c *is an absolute constant.*

Remark 1. It is known that the Lemma is true with $c = 1$ (see [1, Lemma 0–1]).

Remark 2. The estimate (c) together with $c = 1$ can be replaced by

(c′) $|R(z) - \frac{1}{2}| < \frac{1}{2}$ for $z \in F$

which seems more natural and follows from considering $\varphi \circ R$ with $\varphi(z) = (1/2)(1 + (z + x)/(zx + 1))$, $x = -1 + \delta$, $\delta > 0$ small.

Remark 3. Suppose that $\bar{G}$ is extensible relative to $H := \overline{F \setminus \bar{G}}$. Then we see from the definition given in the preceding section that for all $\delta > 0$ sufficiently small we can apply the Lemma with $\varepsilon = \delta/2$ to the sets $\mathring{C} = \mathring{C}(\delta/2)$ instead of G and $C \cup F$ instead of F.

For the resulting rational function R we then obtain

(a′) $|R(z)| < \delta$ for $z \in G \setminus (\partial G \cap \partial H)_\delta$,

(b′) $|R(z) - 1| < \delta$ for $z \in H \setminus (\partial G \cap \partial H)_\delta$,

together with (c′) as above.

Note that (a′), (b′) are estimates of the same type whereas (a) and (b) are not.

Remark 4. Suppose again that $\bar{G}$ is extensible relative to $H := \overline{F \setminus \bar{G}}$ and let for some small $\delta > 0$ a rational function R as above be taken. It follows that if $D \subset F \setminus \bar{\mathring{F}}$ is a fixed set and $\tilde{G} := \bar{G} \cup D$, then there is a continuous function $\tilde{R}$ with $\tilde{R} \equiv R$ on $\bar{\mathring{F}}$ so that (a′), (b′), (c′) are satisfied with $\tilde{R}$, $\tilde{G}$, and $\tilde{H} = \overline{F \setminus \tilde{G}}$ instead of R, G, and H. From the assumption that $\mathbb{C} \setminus F$ has only finitely many components we see from a well known theorem of Mergelyan ([4], cf. [2, p. 110]) that $\tilde{R}$ can be approximated uniformly on F by rational functions. Therefore we can claim the estimates (a′), (b′), (c′) also in the case that G is a subset of F with $\partial \mathring{G} \subset \partial F$, but G not necessarily open, and we note that for this it suffices that $\bar{\mathring{G}}$ be extensible relative to $\overline{\mathring{F} \setminus \bar{G}}$.

Remark 5. Now suppose that $G \subset F$, $\partial \mathring{G} \subset \partial F$, and let $\bar{\mathring{G}}$ be extensible relative to $\overline{\mathring{F} \setminus \bar{G}}$. As remarked above we can find a rational function R_1 which fulfills (a′), (b′), (c′) for the sets G, F, $H = \overline{F \setminus \bar{G}}$.

Let a finite number of points $z_1, ..., z_n \in \bar{G} \setminus (\partial G \cap \partial H)$, $w_1, ..., w_N \in F \setminus \bar{G}$ together with a collection of natural numbers $\nu_1, ..., \nu_n, \mu_1, ..., \mu_N$ be given.

We will prove that if (c′) is replaced by

(c″) $|R(z) - \frac{1}{2}| < \frac{1}{2} + \delta$

we can find a rational function R which fulfills (a′), (b′), (c″) and in addition $R(z_j) = 0$, $R(w_h) = 1$ $(j = 1, ..., n; h = 1, ..., N)$ with multiplicity at least γ_j at z_j and μ_h at w_h.

To obtain this we start with

$$R_2(z) := \prod_{j=1}^{n} (R_1(z) - R_1(z_j))^{\nu_j}.$$

Because (c′) holds for R_1 it follows that

$$|R_2(z)| < 1 + \lambda \qquad \text{for } z \in F$$

for some $\lambda > 0$ depending on the bound δ in (a′), (b′).

Now let $\varphi(w) = (1/2) - (1/2)((w - 1 + \lambda)/(1 - \lambda) \cdot (w - 1))$ $(|w| < 1)$ and

$$R_3(z) := \varphi\left(\frac{1}{1+\lambda} R_2(z)\right) \qquad (z \in F).$$

Then we have $R_3(z_j) = \lambda/2$ $(j = 1, ..., n)$ and we may assume $|R_3(w_h) - 1| < \lambda$ $(h = 1, ..., N)$.

From the construction we obtain $|R_3(z) - 1/2| < 1/2$ for all $z \in F$.

Let

$$R_4(z) = 1 - \prod_{h=1}^{N} \left(1 - \frac{2R_3(z) - \lambda}{2R_3(w_h) - \lambda}\right)^{\mu_h}.$$

A short calculation gives the estimate

$$|R_4(z) - 1| < 1 + \delta \qquad (z \in F)$$

for λ sufficiently small. Similar as above we take a suitable linear transformation ψ which maps $\{|w - 1| < 1\}$ on $\{|\xi - 1/2 - \delta| < 1/2 + \delta\}$ and fulfills $\psi(0) = 0$, $\psi(1) = 1$. Then the function $R = \psi \circ R_4$ has the desired properties for all parameters small enough.

Remark 6. It seems to be an open question if a stronger version of the Lemma of Nersesjan still is true where (a) is replaced by

(a″) $|R(z)| < \varepsilon$ for $z \in G \setminus (F \setminus \bar{G})_\varepsilon$.

If such is the case the assumption of relative extensibility in our theorem above could be replaced by the weaker condition $\partial K_1 \cup \partial K_2 \subset \partial(K_1 \cup K_2)$.

4. A Proof of the Extended Fusion Lemma

Let K_1, K_2 be compact sets and assume without loss of generality that $\mathring{\bar{K}}_1$ is extensible relative to $\mathring{\bar{K}}_2$. Let a further compact set k and rational functions r_1, r_2 be given. Note that the conclusion holds trivially if $r_1 - r_2$ has a pole on $k \cup (K_1 \cap K_2)$. Therefore we may assume that

$$\delta := \sup\{|r_1(w) - r_2(w)| \ |w \in k \cup (K_1 \cap K_2)\} < \infty. \qquad (*)$$

It is sufficient to give the proof in the case $r_2 \equiv 0$ (cf. [2, p. 114]).

From Remark 5 above we obtain a rational function R which has zeros at the poles of $r_1 - r_2$ $(=r_1)$ lying in K_1 and takes the value 1 at the poles of r_1 in K_2.

We will assume that in both cases the order of R is at least the order of the pole of r_1 at these points. From (a′), (b′), (c″), and (*) we conclude that the following estimates can be established:

$$\text{(I)} \quad |R(z) \cdot r_1(z)| \leqslant 2\delta \quad (z \in K_1)$$

$$\text{(II)} \quad |(R(z) - 1)\, r_1(z)| \leqslant 2\delta \quad (z \in K_2).$$

Take a neighborhood V of $K_1 \cup K_2$ such that there are no poles of R in $\bar{V}$. There is some $\mathscr{C}^1$-function $H\colon \mathbb{R}^2 \to \mathbb{C}$ with compact support T and which agrees with R on $\bar{V}$ (cf. [2, p. 107]). We may assume that the boundary of both V and T consists of a finite number of Jordan curves.

For fixed z let $\xi = z + re^{i\varphi}$ and by $\bar{\partial} H$ we understand the derivative of H with respect to $\bar{\xi}$. Then the following estimate for the area integral can easily be established (polar coordinates):

$$\begin{aligned} \frac{1}{\pi} \iint_{\mathbb{R}^2} \frac{|\bar{\partial} H(\xi)|}{|\xi - z|}\, db_\xi &= \frac{1}{\pi} \iint_{T \setminus \bar{V}} \frac{|\bar{\partial} H(\xi)|}{|\xi - z|}\, db_\xi \\ &\leqslant \sup_{T \setminus \bar{V}} |\bar{\partial} H| \cdot \frac{1}{\pi} \cdot 2\pi \operatorname{diam}(T \setminus \bar{V}) <: a \end{aligned} \qquad (**)$$

for a suitable constant a depending on K_1, K_2 only. From (*) we can obtain an open and bounded neighborhood U of k with $|r_1(z)| < 2\delta$ for $z \in U$.

By the Tietze extension theorem we find some continuous function $q\colon \bar{\mathbb{C}} \setminus (\partial V \setminus U) \to \bar{\mathbb{C}}$ with $q \mid V \cup U \equiv r_1 \mid V \cup U$ and q has compact support $T_1 \subset \mathbb{C}$. Moreover we may assume

$$\text{(III)} \quad |q(z)| < 2\delta \text{ for } z \in \bar{\mathbb{C}} \setminus \bar{V}.$$

Let $E := (T \cap T_1) \setminus V$. We may assume that the boundary is a finite union of Jordan curves. As in the proof of the classical Fusion Lemma we now define

$$\begin{aligned} g(z) &:= \frac{1}{\pi} \iint_{\mathbb{R}^2} q(\xi)\, \frac{\bar{\partial} H(\xi)}{\xi - z}\, db_\xi \\ &= \frac{1}{\pi} \iint_E q(\xi)\, \frac{\bar{\partial} H(\xi)}{\xi - z}\, db_\xi \end{aligned}$$

for $z \in \mathbb{C}$. Note that g is holomorphic on $\mathbb{C} \setminus E$.

The function

$$f := -H \cdot q + g$$

is meromorphic on a neighborhood of $K_1 \cup K_2$. From the well-known Pompeiu formula (cf. [2, p. 94]) we conclude

$$f(z) = -\frac{1}{\pi} \iint_E (q(\xi) - q(z)) \frac{\bar{\partial} H(\xi)}{\xi - z} \, db_\xi$$

for all $z \in \mathbb{C}$ with $q(z) \neq \infty$. Therefore f is holomorphic in U. Now by Runge's Theorem we find (cf. [2, p. 116]) a rational function r with

$$|r(z) - r_1(z) - f(z)| \leqslant \delta \qquad (z \in K_1 \cup K_2 \cup k).$$

For r we obtain the following estimates:

1. For $z \in K_1$ we have

$$\begin{aligned} |r(z) - r_1(z)| &\leqslant \delta + |f(z)| \\ &\leqslant \delta + |H(z)\, q(z)| + |g(z)| \\ &= \delta + |R(z)\, r_1(z)| + |g(z)| \\ &\leqslant \delta + 2\delta + 2\delta a \qquad \text{(by (I), (III), (**))} \\ &= (3 + 2a)\, \delta. \end{aligned}$$

2. If $z \in K_2$ we conclude

$$\begin{aligned} |r(z) - r_2(z)| = |r(z)| &\leqslant \delta + |r_1(z) + f(z)| \\ &= \delta + |r_1(z) - R(z)\, r_1(z) + g(z)| \\ &\leqslant \delta + |(R(z) - 1) \cdot r_1(z)| + |g(z)| \\ &\leqslant \delta + 2\delta + 2\delta a \qquad \text{(by (II), (III), (**))} \\ &= (3 + 2a)\, \delta. \end{aligned}$$

3. For $z \in k$ we get with $b := \mathrm{Max}_{z \in \mathbb{C}} |H(z)|$

$$\begin{aligned} |r(z) - r_1(z)| &\leqslant \delta + |f(z)| \\ &\leqslant \delta + b \cdot \delta + 2\delta a \qquad \text{(by (*), (III), (**))} \\ &= (1 + b + 2a)\, \delta \end{aligned}$$

and therefore

$$|r(z) - r_2(z)| \leqslant |r(z) - r_1(z)| + |r_1(z)|$$
$$\leqslant (2 + b + 2a)\,\delta.$$

With $\alpha := \text{Max}\{3 + 2a, 2 + b + 2a\}$ we obtain the desired result.

Remark. In the special case $k = \overline{K_1 \cap K_2}$ we can give a short proof for the theorem. Again we assume that $\overset{\circ}{K}_1$ is extensible relative to $\overset{\circ}{K}_2$. We take the rational function R as in the proof above. Let $r := (r_2 - r_1)\, R + r_1$ where the poles of $r_1 - r_2$ coincide with value 0 (in K_1) respectively 1 (in K_2) of R as above. In analogy to (I) and (II) we get the estimates

$$|r(z) - r_1(z)| = |(r_2(z) - r_1(z))\, R(z)| < 3\delta \qquad (z \in K_1)$$

and

$$|r(z) - r_2(z)| = |(r_2(z) - r_1(z))(R(z) - 1)| < 4\delta \qquad (z \in K_2)$$

so that the conclusion holds with $\alpha = 4$.

Finally we note that in the example sketched in Fig. 2 the extended Fusion Lemma still holds although K_1 is not extensible relative to K_2, but K_2 is extensible relative to K_1.

Acknowledgment

The author is grateful to the referee for helpful comments.

References

1. A. Boivin, Carleman approximation on Riemann surfaces, *Math. Ann.* **275** (1986), 57–70.
2. D. Gaier, "Vorlesungen über Approximation im Komplexen," Birkhäuser, Basel/Boston/Stuttgart, 1980 (an English translation appeared recently).
3. D. Gaier, Remarks about Alice Roth's Fusion Lemma, *J. Approx. Theory* **37** (1983), 246–250.
4. S. N. Mergelyan, Uniform approximations to functions of a complex variable (Russian), *Uspekhi Mat. Nauk* **7**, No. 2 (48) (1952), 31–122 (Translation, *Amer. Math. Soc.* **3** (1962), 294–391).
5. A. A. Nersesjan, Carleman sets (Russian), *Izv. Akad. Nauk Armyan. SSR Ser. Mat.* **6** (1971), 466–471.
6. A. Roth, Uniform and tangential approximation by meromorphic functions on closed sets, *Canad. J. Math.* **28** (1976), 104–111.
7. G. Schmieder, Über die simultane Approximation zweier meromorpher Funktionen, *Sitzungsber. Bayr. Akad. Wiss.* (1983), 133–139.
8. G. Schmieder, Approximation auf abgeschlossenen Teilen Riemannscher Flächen, *Manuscripta Math.* **46** (1984), 165–192.